Geometric Optimal Control with Applications

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Contents

Chapter 3

Calculus of Variations

This section deals with the Calculus of Variations.

3.1 Statement of the Problem in the Holonomic Case

We consider the set $\mathcal{C}$ of all curves $x : [t_0, t_1] \to \mathbb{R}^n$ of class C^2, the intitial and final times t_0, t_1 being not fixed and the problem of minimizing a functional on $\mathcal{C}$:

$$C(x) = \int_{t_0}^{t_1} L(t, x(t), \dot{x}(t))dt$$

where L is C^2. Moreover, we impose extremeties conditions: $x(t_0) \in M_0, x(t_1) \in M_1$ where M_0, M_1 are C^1-submanifolds of $\mathbb{R}^n$. The distance between the curves $x(t), x^*(t)$ is

$$\rho(x, x*) = \max_t \|x(t) - x^*(t)\| + \max_t \|\dot{x}(t) - \dot{x}^*(t)\| + d(P_0, P_0^*) + d(P_1, P_1^*)$$

where $P_0 = (t_0, x_0)$ and $P_1 = (t_1, x_1)$ and $\| \cdot \|$ is any norm on $\mathbb{R}^n$ and d is the usual distance mapping on $\mathbb{R}^{n+1}$. The two curves $x(\cdot), c^*(\cdot)$ being not defined on the same interval they are by convention C^2-extended on the union of both intervals.

Proposition 1. ***(Fundamental formula of the classical calculus of variations)*** *We adopt the standard notation of classical calculus of variations, see [12]. Let $\gamma(\cdot)$ be a reference curve with extremeties $(t_0, x_0), (t_1, x_1)$ and let $\underline{\gamma}(\cdot)$ be any curve with extremeties $(t_0 + \delta t_0, x_0 + \delta x_0), (t_1 + \delta t_1, x_1 + \delta x_1)$. We denote by $h(\cdot)$ the variation: $h(t) = \underline{\gamma}(t) - \gamma(t)$. Then, if we set $\Delta C = C(\underline{\gamma}) - C(\gamma)$, we have*

$$\Delta C = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right)_{|\gamma} .h(t)dt + \left[\frac{\partial L}{\partial \dot{x}_{|\gamma}} .\delta x \right]_{t_0}^{t_1} + \left[(L - \frac{\partial L}{\partial \dot{x}} .\dot{x})_{|\gamma} \delta t \right]_{t_0}^{t_1} + o(\rho(\underline{\gamma}, \gamma)) \tag{3.1}$$

where . denotes the scalar product in $\mathbb{R}^n$.

Proof. We write

$$\begin{aligned}
\Delta C &= \int_{t_0+\delta t_0}^{t_1+\delta t_1} L(t, \gamma(t) + h(t), \dot{\gamma}(t) + \dot{h}(t))dt - \int_{t_0}^{t_1} L(t, \gamma(t), \dot{\gamma}(t))dt \\
&= \int_{t_0}^{t_1} L(t, \gamma(t) + h(t), \dot{\gamma}(t) + \dot{h}(t))dt - \int_{t_0}^{t_1} L(t, \gamma(t), \dot{\gamma}(t))dt \\
&+ \int_{t_1}^{t_1+\delta t_1} L(t, \gamma(t) + h(t), \dot{\gamma}(t) + \dot{h}(t))dt - \int_{t_0}^{t_0+\delta t_0} L(t, \gamma(t) + h(t), \dot{\gamma}(t) + \dot{h}(t))dt
\end{aligned}$$

We develop this expression using the Taylor expansions keeping only the linear terms in $h, \dot{h}, \delta x, \delta t$. We get

$$\Delta C = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x_{|\gamma}} .h(t) + \frac{\partial L}{\partial \dot{x}_{|\gamma}} .\dot{h}(t) \right) + [L(t, \gamma, \dot{\gamma})\delta t]_{t_0}^{t_1} + o(h, \dot{h}, \delta t).$$

The derivative of the variation $\dot{h}$ is depending on h, integrating by parts we obtain

$$\Delta C \sim \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)_{|\gamma} .h(t)dt + \left[\frac{\partial L}{\partial \dot{x}_{|\gamma}} .h(t)\right]_{t_0}^{t_1} + \left[L_{|\gamma}\delta t\right]_{t_0}^{t_1}$$

We observe that $h, \delta x, \delta t$ are not dependent at the extremeties and we have for $t = t_0$ or $t = t_1$ the relation

$$h(t + \delta t) \sim h(t) \sim \delta x - \dot{x}\delta t$$

Hence, we obtain the following approximation:

$$\Delta C \sim \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)_{|\gamma} .h(t)dt + \left[\frac{\partial L}{\partial \dot{x}_{|\gamma}} .\delta x\right]_{t_0}^{t_1} + \left[(L - \frac{\partial L}{\partial \dot{x}}\dot{x})_{|\gamma}\delta t\right]_{t_0}^{t_1}$$

where all the quantities are evaluated along the reference trajectory $\gamma(\cdot)$. In this formula $h, \delta x, \delta t$ can be taken independent because in the integral the values $h(t_0), h(t_1)$ do not play any special role. □

From 3.1, we deduce that the standard first-order necessary conditions of the calculus of variations.

Corollary 1. *Let us consider the minimization problem where the extremities $(t_0, x_0), (t_1, x_1)$ are fixed. Then a minimizer $\gamma(\cdot)$ satisfies the Euler-Lagrange equation*

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{|\gamma}} = 0 \tag{3.2}$$

Proof. Since the extremities are fixed we set in (3.1) $\delta x = 0$ and $\delta t = 0$ at $t = t_0$ and $t = t_1$. Hence

$$\Delta C = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)_{|\gamma} .h(t)dt + o(h, \dot{h})$$

for each variation $h(\cdot)$ defined on $[t_0, t_1]$ such that $h(t_0) = h(t_1) = 0$. If $\gamma(\cdot)$ is a minimizer, we must have $\Delta C \geq 0$ for each $h(\cdot)$ and clearly by linearity, we have

$$\int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)_{|\gamma} .h(t)dt = 0$$

for each $h(\cdot)$. Since the mapping $t \mapsto (\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}})_{|\gamma}$ is continuous, it must be identically zero along $\gamma(\cdot)$ and the Ealer-Lagrange equation 3.2 is satisfied. □

3.1.1 Hamiltonian Equations

The Hamiltonian formalism, which is the natural formalism to deal with the maximum principle, appears in the classical calculus of variations via the **Legendre transformation**.

Definition 1. *The Legendre transformation is defined by*

$$p = \frac{\partial L}{\partial \dot{x}}(t, x, \dot{x}) \tag{3.3}$$

and if the mapping $\varphi : (x, \dot{x}) \mapsto (x, p)$ is a diffeomorphism we can introduce the Hamiltonian:

$$H : (t, x, p) \mapsto p.\dot{x} - L(t, x, p). \tag{3.4}$$

Proposition 2. *The formula (3.1) takes the form*

$$\Delta C \sim \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)_{|\gamma} .h(t)dt + \left[p\delta x - H\delta t\right]_{t_0}^{t_1} \tag{3.5}$$

and if $\gamma(\cdot)$ is a minimizer it satisfies the Euler-Lagrange equation in the Hamiltonian form

$$\dot{x}(t) = \frac{\partial H}{\partial p}(t, x(t), p(t)), \qquad \dot{p}(t) = -\frac{\partial H}{\partial x}(t, x(t), p(t)) \tag{3.6}$$

3.1.2 Hamilton-Jacobi-Bellman Equation

Definition 2. *A solution of the Euler-Lagrange equation is called an extremal. Let* $P_0 = (t_0, x_0)$ *and* $P_1 = (t_1, x_1)$*. The Hamilton-Jacobi-Bellman (HJB) function is the multivalued function defined by*

$$S(P_0, P_1) = \int_{t_0}^{t_1} L(t, \gamma(t), \dot{\gamma}(t))dt$$

where $\gamma(\cdot)$ *is any extremal with fixed extremities* x_0, x_1*. If* $\gamma(\cdot)$ *is a minimizer, it is called the value function.*

Proposition 3. *Assume that for each* P_0, P_1 *there exists a unique extremal joining* P_0 *to* P_1 *and suppose that the HJB function is* C^1*. Let* P_0 *be fixed and let* $\bar{S} : P \mapsto S(P_0, P)$*. Then* $\bar{S}$ *is a solution of the Hamilton-Jacobi-Bellman equation*

$$\frac{\partial S}{\partial t}(P_0, P) + H(t, x, \frac{\partial S}{\partial x}) = 0 \tag{3.7}$$

Proof. Let $P = (t, x)$ and $P + \delta P = (t + \delta t, x + \delta x)$. Denote by $\gamma(\cdot)$ the extremal joining P_0 to $P + \delta P$. We have

$$\Delta\bar{S} = \bar{S}(t + dt, x + dx) - \bar{S}(t, x) = C(\bar{\gamma}) - C(\gamma)$$

and from (2) it follows that:

$$\Delta\bar{S} = \Delta C \sim \int_{t_0}^{t} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)_{|\gamma} .h(t)dt + \Big[p\delta x - H\delta t\Big]_{t_0}^{t},$$

where $h(\cdot) = \bar{\gamma}(\cdot) - \gamma(\cdot)$. Since $\gamma(\cdot)$ is a solution of the Euler-Lagrange equation, the integral is zero and

$$\Delta\bar{S} = \Delta C \sim \Big[p\delta x - H\delta t\Big]_{t_0}^{t}$$

In other words, we have

$$d\bar{S} = pdx - Hdt.$$

Identifying, we obtain

$$\frac{\partial \bar{S}}{\partial t} = -H, \qquad \frac{\partial \bar{S}}{\partial x} = p. \tag{3.8}$$

Hence we get the HJB equation. Moreover p is the gradient to the level sets $\{x \in \mathbb{R}^n; \bar{S}(t, x) = c\}$. □

3.1.3 Euler-Lagrange Equations and Characteristics of the HJB Equation

Under some extra regularity conditions, the extremals are the characteristics of the HJB equation. Indeed, let $u(\cdot)$ be a solution of the HJB equation. Hence we can write (3.7) as

$$F(t, x, \frac{\partial S}{\partial t}, \frac{\partial S}{\partial x}) = \frac{\partial S}{\partial t} + H(t, x, \frac{\partial S}{\partial x}) = 0$$

and let us assume the map F to be C^2. Introduce $p = \frac{\partial S}{\partial x}, T = \frac{\partial S}{\partial t}$ and $z = S(t, x)$. Then, according to [19], the characteristic curves parameterized by s are solutions of:

$$\begin{aligned}
\frac{dx}{ds} &= \frac{\partial F}{\partial p} = \frac{\partial H}{\partial p} \\
\frac{dp}{ds} &= -\frac{\partial F}{\partial x} - \frac{\partial F}{\partial z}p = -\frac{\partial H}{\partial x} \\
\frac{dz}{ds} &= p\frac{\partial F}{\partial p} + T = p\frac{\partial H}{\partial p} - H \\
\frac{dt}{ds} &= \frac{\partial F}{\partial T} = 1 \\
\frac{dT}{ds} &= -\frac{\partial F}{\partial t} - \frac{\partial F}{\partial z}p = -\frac{\partial H}{\partial t}
\end{aligned} \tag{3.9}$$

In particular since $\frac{dt}{ds} = 1$, we deduce that

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

which is the Hamiltonian form of the Euler-Lagrange equation.

3.1.4 Second Order Conditions

The Euler-Lagrange equation has been derived using the linear terms in the Taylor expansion of ΔC. Using the quadratic terms we can get necessary and sufficient second order condition. For the sake of simplicity, from now on we assume that the curves $t \mapsto x(t)$ **belongs to** $\mathbb{R}$, and we consider the problem with fixed extremities: $x(t_0) = x_0, x(t_1 = x_1)$. If the map L is taken C^3, the second derivative is computed as follows:

$$\begin{aligned}\Delta C &= \int_{t_0}^{t_1} \Big(L(t), \gamma(t) + h(t), \dot{\gamma}(t) + \dot{h}(t) - L(t, \gamma(t), \dot{\gamma(t)}) \Big)\, dt \\ &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right)_{|_\gamma} .h(t)dt + \frac{1}{2}\int_{t_0}^{t_1} \Big((\frac{\partial^2 L}{\partial x^2})_{|_\gamma} h^2(t) + 2(\frac{\partial^2 L}{\partial x \partial \dot{x}})_{|_\gamma} h(t)\dot{h}(t) \\ &+ (\frac{\partial^2 L}{\partial \dot{x}^2})_{|_\gamma} \dot{h}^2(t) \Big) dt + o(h, \dot{h})_2\end{aligned}$$

If $\gamma(t)$ is an extremal, the first integral is zero and the second integral corresponds to the **intrinsic second-order derivative** $\delta^2 C$, that is:

$$\delta^2 C = \frac{1}{2}\int_{t_0}^{t_1} \Big((\frac{\partial^2 L}{\partial x^2})_{|_\gamma} h^2(t) + 2(\frac{\partial^2 L}{\partial x \partial \dot{x}})_{|_\gamma} h(t)\dot{h}(t) + (\frac{\partial^2 L}{\partial \dot{x}^2})_{|_\gamma} \dot{h}^2(t) \Big) dt \tag{3.10}$$

Using $h(t_0) = h(t_1) = 0$, it can be written after an integration by parts as

$$\delta^2 C = \int_{t_0}^{t_1} \Big(P(t)\dot{h}^2(t) + Q(t)h^2(t) \Big) dt \tag{3.11}$$

where

$$P = \frac{1}{2}(\frac{\partial^2 L}{\partial \dot{x}^2})_{|_\gamma}, \qquad Q = \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^2} - \frac{d}{dt}\frac{\partial^2 L}{\partial x \partial \dot{x}} \right)_{|_\gamma}.$$

Using the fact that in the integral (3.11) the term $P\dot{h}^2$ is dominating [12], we get the following proposition.

Proposition 4. *If $\gamma(\cdot)$ is a minimizing curve for the fixed extremities problem then it must satisfy the Legendre condition:*

$$(\frac{\partial^2 L}{\partial \dot{x}^2})_{|_\gamma} \geq 0. \tag{3.12}$$

3.1.5 The Accessory Problem and the Jacobi Equation

The intrinsic second-order derivative is given by

$$\delta^2 C = \int_{t_0}^{t_1} \Big(P(t)\dot{h}^2(t) + Q(t)h^2(t) \Big) dt, \qquad h(t_0) = h(t_1) = 0,$$

where P, Q are as above. We write

$$\delta^2 C = \int_{t_0}^{t_1} \Big((P(t)\dot{h}(t))\dot{h}(t) + (Q(t)h(t))h(t) \Big) dt$$

and integrating by parts using $h(t_0) = h(t_1) = 0$, we obtain

$$\delta^2 C = \int_{t_0}^{t_1} \Big(Q(t)h(t) - \frac{d}{dt}(P(t)\dot{h}(t))\Big)h(t)dt$$

Let us introduce the linear operator $D : h \mapsto Qh - \frac{d}{dt}(P\dot{h})$. Hence, we can write

$$\delta^2 C = (Dh, h) \tag{3.13}$$

where $(,)$ is the usual scalar product on $L^2([t_0, t_1])$. The linear operator D is called the **Euler-Lagrange operator.**

Definition 3. *From (3.13), $\delta^2 C$ is a quadratic operator on the set $\mathcal{C}_0$ of C^2-curves $h : [t_0, t_1] \to \mathbb{R}$ satisfying $h(t_0) = h(t_1) = 0, h \neq 0$. Rather to study $\delta^C > 0$ for each $h(\cdot) \in \mathcal{C}_0$ we can study the so-called accessory problem: $\min_{h\in\mathcal{C}_0} \delta^2 C$.*

Definition 4. *The Euler-Lagrange equation corresponding to the accessory problem is called the Jacobi equation and is given by*

$$Dh = 0 \tag{3.14}$$

where D is the Euler-Lagrange operator: $Dh = Qh - \frac{d}{dt}(P\dot{h})$. It is a second-order linear differential operator.

Definition 5. *The strong Legendre condition is $P > 0$ where $P = \frac{1}{2}(\frac{\partial^2 L}{\partial \dot{x}^2})_{|\gamma}$. If this condition is satisfied, the operator D is said to be nonsingular.*

3.1.6 Conjugate Point and Local Morse Theory

See also [14], [27].

Definition 6. *Let $\gamma(\cdot)$ be an extremal. A solution $J(\cdot) \in \mathcal{C}_0$ of $DJ = 0$ on $[t_0, t_1]$ is called a Jacobi curve. If there exists a Jacobi curve along $\gamma(\cdot)$ on $[t_0, t_1]$ the point $\gamma(t_1)$ is said to be conjugate to $\gamma(t_0)$.*

Theorem 1. ***(Local Morse theory [27]).*** *Let t_0 be fixed and let us consider the Euler-Lagrange operator (indexed by $t > t_0$) D^t defined on the set $\mathcal{C}_0^t$ of curves on $[t_0, t]$ satisfying $h(t_0) = h(t_1) = 0$. By definition, a Jacobi curve on $[t_0, t]$ corresponds to an eigenvector J^t associated to an eigenvalue $\lambda^t = 0$ of D^t. If the strong Legendre condition is satisfied along an extremal $\gamma : [t_0, t] \to \mathbb{R}^n$, we have a precise description of the spectrum of D^t as follows. There exists $t_0 < t_1 < \cdots < t_s < T$ such that each $\gamma(t_i)$ is conjugate to $\gamma(t_0)$. If n_i corresponds to the dimension of the set of the Jacobi curves J^{t_i} associated to the conjugate point $\gamma(t_i)$, then for any $\tilde{T}$ such that $t_0 < t_1 < \cdots < t_k < \tilde{T} < t_{k+1} < \cdots < T$ we have the identity*

$$n^-_{\tilde{T}} = \sum_{i=1}^{k} n_i \tag{3.15}$$

where $n^-_{\tilde{T}} = dim\{$linear space of eigenvectors of $D^{\tilde{T}}$ corresponding to strictly negative eigenvalues$\}$. In particular if $\tilde{T} > t_1$ we have

$$\min_{h\in\mathcal{C}_0} \int_{t_0}^{\tilde{T}} (Q(t)h^2(t) + P(t)\dot{h}^2(t))dt = -\infty \tag{3.16}$$

3.1.7 Scalar Riccati Equation

Definition 7. *The quadratic differential equation*

$$P(t)(Q(t) + \dot{w}(t)) = w^2(t) \tag{3.17}$$

is called the scalar Riccati equation.

Its connections with the problem is the following. Assume $P > 0$ on $[t_0, t_1]$ and assume that there exists a solution $u(\cdot)$ of the Jacobi equation such that this solution **does not vanish** on the interval $[t_0, t_1]$.

Let $h(\cdot)$ be any C^2-function such that $h(t_0) = h(t_1) = 0$. Then

$$\int_{t_0}^{t_1} d(w(t)h^2(t))dt = 0$$

and

$$\delta^2 C = \int_{t_0}^{t_1} \left((P\dot{h}^2 + Qh^2) + d(wh^2)\right) dt = \int_{t_0}^{t_1} \left(P\dot{h}^2 + 2wh\dot{h} + (Q + \dot{w})h^2\right) dt.$$

If $w(\cdot)$ is a solution of the Riccati equation, the previous expression can be written as

$$\delta^2 C = \int_{t_0}^{t_1} P(t) \left(\dot{h} + \frac{w(t)}{P(t)} h(t)\right)^2 dt.$$

Hence

$$\delta^2 C = \int_{t_0}^{t_1} P(t)\varphi^2(t)dt$$

where $\varphi(t) = \dot{h} + \frac{w(t)h(t)}{P(t)}$. Now observe that if we set $w(t) = -\frac{\dot{u}(t)}{u(t)} P(t)$ where $u(\cdot)$ is nonvanishing on $[t_0, t_1]$, then we get that $u(\cdot)$ is a solution of the Jacobi equation:

$$Q(t)u(t) - \frac{d}{dt}(P(t)\dot{u}(t)) = 0$$

and

$$\dot{h}(t) + \frac{w(t)h(t)}{P(t)} = \frac{\dot{h}(t)u(t) - h(t)\dot{u}(t)}{u(t)}.$$

Hence $\varphi(t) \equiv 0$ is equivalent to

$$\dot{h}(t)u(t) - h(t)\dot{u}(t) = 0.$$

This is possible if and only if $h(\cdot) = Cu(\cdot)$ where C is a constant. It contradicts the fact that $u(\cdot)$ does not vanish on $[t_0, t_1]$ and that $h(t_0) = h(t_1) = 0$ if $h \neq 0$. Hence $\varphi \neq 0$ unless $h \equiv 0$ and

$$\delta^2 C = \int_{t_0}^{t_1} P(t)\varphi^2(t)dt$$

is nonzero for each $h(\cdot) \in \mathcal{C}_0$ and $\delta^2 C > 0$.

3.1.8 Local C^0 Minimizer - Extremal Field - Hilbert Invariant Integral

Definition 8. *Consider the time-minimal problem with fixed extremities: $(t, x_0), (t_1, x_1) \in \mathbb{R}^{n+1}$. Let $\gamma(\cdot)$ be a reference trajectory. It is called a C^0-minimizer if it is a local minimum for the C^0-topology:*

$$d(x, x^*) = \max_{t \in [t_0, t_1]} \|x(t) - x^*(t)\|$$

To obtain C^0-sufficient optimality conditions we use the concept of **extremal** (or **Mayer field**).

Definition 9. *Let $\gamma : [t_0, t_1] \to \mathbb{R}^n$ be a reference extremal issued from x_0 at $t = t_0 : \gamma(t_0) = x_0$. An extremal field is a mapping $\phi : (\alpha, t) \to \mathbb{R}^{n+1}, \alpha \in D =$ parameter space $\subset \mathbb{R}^n$ such that:*

1. *$\phi(\alpha_0, t) = (t, \gamma(t))$ is the reference extremal and $\{\phi(\alpha, \cdot); \alpha \in D\}$ is a family $\mathcal{F}$ of extremals;*
2. *the image of ϕ denoted $\mathcal{T}$ is a tubular neighborhood of $\gamma(\cdot)$ and through each point (t, x) there passes a unique extremal of $\mathcal{F}$ whose derivative is denoted by $u(t, x)$.*
3. *The field is formed by extremals, starting at $t_0 - \varepsilon$ from a single point $\gamma(t_0 - \varepsilon)$ for $\varepsilon > 0$ small enough.*

We assume that $t \mapsto u(t,x)$ is C^1 and we use the following notations:

$$\hat{p} = \frac{\partial L}{\partial \dot{x}}_{|_{\dot{x}=u(t,x)}}, \quad \hat{L} = L_{|_{\dot{x}=u(t,x)}}, \quad \hat{H} = H_{p=\hat{p}} \tag{3.18}$$

where $H = p.\dot{x} - L$ is the Hamiltonian.

Lemma 1. *The following relations hold:*

$$\frac{\partial \hat{p}_i}{\partial x_k} = \frac{\partial \hat{p}_k}{\partial x_i}, \qquad \frac{\partial \hat{H}}{\partial x_i} = -\frac{\partial \hat{p}_i}{\partial t}.$$

In particular the one form $\hat{\omega} = -\hat{H}dt + \hat{p}dx$, called Hilbert-Cartan form, is closed.

Theorem 2. ***(Hilbert invariant integral theorem).*** *The integral*

$$\int_\Gamma -\hat{H}dt + \hat{p}dx$$

is independent of the curve $\Gamma(\cdot)$ on $\mathcal{T}$. Moreover if $\Gamma(\cdot)$ is an extremal of $\mathcal{F}$ it is given by $\int_\Gamma Ldt$.

Proof. The first assertion is a consequence of the fact that the form $\hat{\omega}$ is closed. Moreover if $\Gamma(\cdot)$ is an extremal we have $\frac{dx}{dt} = u(t,x)$ and $-\hat{H}dt + \hat{p}dx = (\hat{L} - \hat{p}.u(t,x))dt + \hat{p}dx$. □

Remark. The resolution of $\hat{\omega} = dS$ on the domain where $S : (t,x) \to \mathbb{R}$ is a smooth function, is equivalent to solve the Hamilton-Jacobi equation.

Corollary 2. *Let $\gamma(\cdot)$ be the reference extremal with extremities $(t_0,x_0),(t_1,x_1)$ and let $\Gamma(\cdot)$ be any curve of $\mathcal{T}$ with the same extremities. We define by E the excess Weierstrass mapping:*

$$E(t,x,z,w) = L(t,x,w) - L(t,x,z) - (w-z)\frac{\partial L}{\partial \dot{x}}(t,x,z),$$

$z = u(t,x,(t,x) \in \mathcal{T})$. Then, if $E \geq 0$ we have that $\gamma(\cdot)$ is a C^0 minimizer on $\mathcal{T}$.

Proof. We have

$$\int_\gamma L(t,x,\dot{x})dt = \int_\Gamma (\hat{L} - \hat{p}.u(t,x))dt + \hat{p}dx,$$

hence

$$\begin{aligned} \Delta C &= \int_\Gamma L(t,x,\dot{x})dt - \int_\gamma L(t,x,\dot{x})dt \\ &= \int_\Gamma \Big((L - \hat{L}) - (\dot{x} - u(t,x).\hat{p}) \Big)\, dt \\ &= \int_\Gamma E(t,x,u(t,x),\dot{x})dt \end{aligned}$$

This proves the assertion. □

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